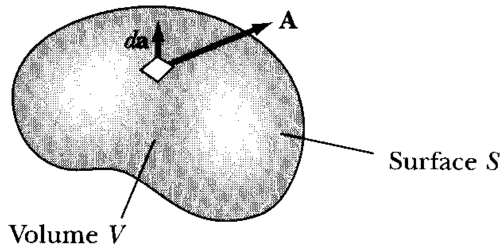

FIGURE 1-23 The differential $d\mathbf{a}$ is an element of area on a surface S that surrounds a closed volume V.

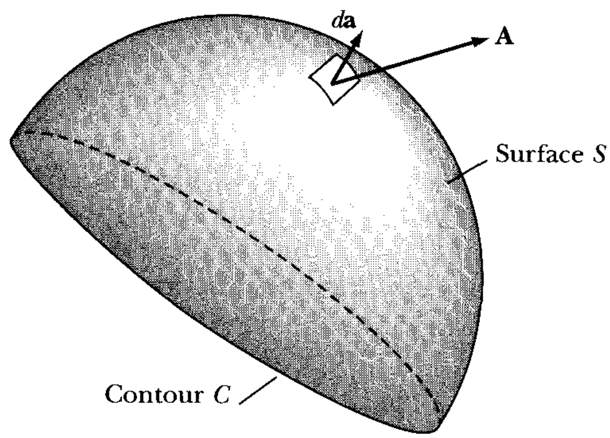

FIGURE 1-24 A contour path C defines an open surface S. A line integral around the path C and a surface integral over the surface S is required for Stokes's theorem.

is hoped, a simpler line integral (one dimensional). Both Gauss's and Stokes's theorems have wide application in vector calculus. In addition to mechanics, they are also useful in electromagnetic applications and in potential theory.

PROBLEMS

1-1. Find the transformation matrix that rotates the axis x_3 of a rectangular coordinate system 45° toward x_1 around the x_2-axis.

1-2. Prove Equations 1.10 and 1.11 from trigonometric considerations.

1-3. Find the transformation matrix that rotates a rectangular coordinate system through an angle of 120° about an axis making equal angles with the original three coordinate axes.

1-4. Show
(a) $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$ (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

1-5. Show by direct expansion that $|\boldsymbol{\lambda}|^2 = 1$. For simplicity, take $\boldsymbol{\lambda}$ to be a two-dimensional orthogonal transformation matrix.

1-6. Show that Equation 1.15 can be obtained by using the requirement that the transformation leaves unchanged the length of a line segment.

1-7. Consider a unit cube with one corner at the origin and three adjacent sides lying along the three axes of a rectangular coordinate system. Find the vectors describing the diagonals of the cube. What is the angle between any pair of diagonals?

1-8. Let **A** be a vector from the origin to a point P fixed in space. Let **r** be a vector from the origin to a variable point $Q(x_1, x_2, x_3)$. Show that

$$\mathbf{A} \cdot \mathbf{r} = A^2$$

is the equation of a plane perpendicular to **A** and passing through the point P.

1-9. For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

(a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$ (b) component of **B** along **A** (c) angle between **A** and **B**
(d) $\mathbf{A} \times \mathbf{B}$ (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

1-10. A particle moves in a plane elliptical orbit described by the position vector

$$\mathbf{r} = 2b \sin \omega t\, \mathbf{i} + b \cos \omega t\, \mathbf{j}$$

(a) Find **v**, **a**, and the particle speed.
(b) What is the angle between **v** and **a** at time $t = \pi/2\omega$?

1-11. Show that the *triple scalar product* $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ can be written as

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Show also that the product is unaffected by an interchange of the scalar and vector product operations or by a change in the order of **A**, **B**, **C**, as long as they are in cyclic order; that is,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}, \qquad \text{etc.}$$

We may therefore use the notation **ABC** to denote the triple scalar product. Finally, give a geometric interpretation of **ABC** by computing the volume of the parallelepiped defined by the three vectors **A**, **B**, **C**.

1-12. Let **a**, **b**, **c** be three constant vectors drawn from the origin to the points A, B, C. What is the distance from the origin to the plane defined by the points A, B, C? What is the area of the triangle ABC?

1-13. **X** is an unknown vector satisfying the following relations involving the known vectors **A** and **B** and the scalar ϕ,

$$\mathbf{A} \times \mathbf{X} = \mathbf{B}, \quad \mathbf{A} \cdot \mathbf{X} = \phi.$$

Express **X** in terms of **A**, **B**, ϕ, and the magnitude of **A**.

1-14. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$$

Find the following
(a) $|\mathbf{AB}|$ (b) $\mathbf{AC}$ (c) $\mathbf{ABC}$ (d) $\mathbf{AB} - \mathbf{B}^t\mathbf{A}^t$

1-15. Find the values of α needed to make the following transformation orthogonal.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}$$

1-16. What surface is represented by $\mathbf{r} \cdot \mathbf{a} =$ const. that is described if $\mathbf{a}$ is a vector of constant magnitude and direction from the origin and $\mathbf{r}$ is the position vector to the point $P(x_1, x_2, x_3)$ on the surface?

1-17. Obtain the cosine law of plane trigonometry by interpreting the product $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$ and the expansion of the product.

1-18. Obtain the sine law of plane trigonometry by interpreting the product $\mathbf{A} \times \mathbf{B}$ and the alternate representation $(\mathbf{A} - \mathbf{B}) \times \mathbf{B}$.

1-19. Derive the following expressions by using vector algebra:
(a) $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$
(b) $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

1-20. Show that

(a) $\sum_{i,j} \varepsilon_{ijk}\, \delta_{ij} = 0$ (b) $\sum_{j,k} \varepsilon_{ijk}\, \varepsilon_{ljk} = 2\delta_{il}$ (c) $\sum_{i,j,k} \varepsilon_{ijk}\, \varepsilon_{ijk} = 6$

1-21. Show (see also Problem 1-11) that

$$\mathbf{ABC} = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j C_k$$

1-22. Evaluate the sum $\sum_k \varepsilon_{ijk}\varepsilon_{lmk}$ (which contains 3 terms) by considering the result for all possible combinations of i, j, l, m; that is,
(a) $i = j$ (b) $i = l$ (c) $i = m$ (d) $j = l$ (e) $j = m$ (f) $l = m$
(g) $i \neq l$ or m (h) $j \neq l$ or m

Show that

$$\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

and then use this result to prove

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

1-23. Use the ε_{ijk} notation and derive the identity

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D}$$

1-24. Let **A** be an arbitrary vector, and let **e** be a unit vector in some fixed direction. Show that

$$\mathbf{A} = \mathbf{e}(\mathbf{A} \cdot \mathbf{e}) + \mathbf{e} \times (\mathbf{A} \times \mathbf{e})$$

What is the geometrical significance of each of the two terms of the expansion?

1-25. Find the components of the acceleration vector **a** in spherical coordinates.

1-26. A particle moves with $v =$ const. along the curve $r = k(1 + \cos\theta)$ (a *cardioid*). Find $\ddot{\mathbf{r}} \cdot \mathbf{e}_r = \mathbf{a} \cdot \mathbf{e}_r, |\mathbf{a}|$, and $\dot{\theta}$.

1-27. If **r** and $\dot{\mathbf{r}} = \mathbf{v}$ are both explicit functions of time, show that

$$\frac{d}{dt}[\mathbf{r} \times (\mathbf{v} \times \mathbf{r})] = r^2\mathbf{a} + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} - (v^2 + \mathbf{r} \cdot \mathbf{a})\mathbf{r}$$

1-28. Show that

$$\nabla(\ln|\mathbf{r}|) = \frac{\mathbf{r}}{r^2}$$

1-29. Find the angle between the surfaces defined by $r^2 = 9$ and $x + y + z^2 = 1$ at the point $(2, -2, 1)$.

1-30. Show that $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

1-31. Show that

(a) $\nabla r^n = nr^{(n-2)}\mathbf{r}$ (b) $\nabla f(r) = \dfrac{\mathbf{r}}{r}\dfrac{df}{dr}$ (c) $\nabla^2(\ln r) = \dfrac{1}{r^2}$

1-32. Show that

$$\int (2a\mathbf{r} \cdot \dot{\mathbf{r}} + 2b\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\,dt = a\mathbf{r}^2 + b\dot{r}^2 + \text{const.}$$

where **r** is the vector from the origin to the point (x_1, x_2, x_3). The quantities r and $\dot{r}$ are the magnitudes of the vectors **r** and $\dot{\mathbf{r}}$, respectively, and a and b are constants.

1-33. Show that

$$\int\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right)dt = \frac{\mathbf{r}}{r} + \mathbf{C}$$

where **C** is a constant vector.

1-34. Evaluate the integral

$$\int \mathbf{A} \times \ddot{\mathbf{A}} dt$$

1-35. Show that the volume common to the intersecting cylinders defined by $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $V = 16a^3/3$.

1-36. Find the value of the integral $\int_S \mathbf{A} \cdot d\mathbf{a}$, where $\mathbf{A} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ and S is the closed surface defined by the cylinder $c^2 = x^2 + y^2$. The top and bottom of the cylinder are at $z = d$ and 0, respectively.

1-37. Find the value of the integral $\int_S \mathbf{A} \cdot d\mathbf{a}$, where $\mathbf{A} = (x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and the surface S is defined by the sphere $R^2 = x^2 + y^2 + z^2$. Do the integral directly and also by using Gauss's theorem.

1-38. Find the value of the integral $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$ if the vector $\mathbf{A} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the surface defined by the paraboloid $z = 1 - x^2 - y^2$, where $z \geq 0$.

1-39. A plane passes through the three points $(x, y, z) = (1, 0, 0), (0, 2, 0), (0, 0, 3)$. **(a)** Find a unit vector perpendicular to the plane. **(b)** Find the distance from the point $(1, 1, 1)$ to the closest point of the plane and the coordinates of the closest point.

1-40. The height of a hill in meters is given by $z = 2xy - 3x^2 - 4y^2 - 18x + 28y + 12$, where x is the distance east and y is the distance north of the origin. **(a)** Where is the top of the hill and how high is it? **(b)** How steep is the hill at $x = y = 1$, that is, what is the angle between a vector perpendicular to the hill and the z axis? **(c)** In which compass direction is the slope at $x = y = 1$ steepest?

1-41. For what values of a are the vectors $\mathbf{A} = 2a\mathbf{i} - 2\mathbf{j} + a\mathbf{k}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 2\mathbf{k}$ perpendicular?