Newtonian mechanics is therefore subject to fundamental limitations when small distances or high velocities are encountered. Difficulties with Newtonian mechanics may also occur when massive objects or enormous distances are involved. A practical limitation also occurs when the number of bodies constituting the system is large. In Chapter 8, we see that we cannot obtain a general solution in closed form for the motion of a system of more than two interacting bodies even for the relatively simple case of gravitational interaction. To calculate the motion in a three-body system, we must resort to a numerical approximation procedure. Although such a method is in principle capable of any desired accuracy, the labor involved is considerable. The motion in even more complex systems (for example, the system composed of all the major objects in the solar system) can likewise be computed, but the procedure rapidly becomes too unwieldy to be of much use for any larger system. To calculate the motion of the individual molecules in, say, a cubic centimeter of gas containing $\approx 10^{19}$ molecules is clearly out of the question. A successful method of calculating the *average* properties of such systems was developed in the latter part of the nineteenth century by Boltzmann, Maxwell, Gibbs, Liouville, and others. These procedures allowed the dynamics of systems to be calculated from probability theory, and a *statistical mechanics* was evolved. Some comments regarding the formulation of statistical concepts in mechanics are found in Section 7.13.

PROBLEMS

2-1. Suppose that the force acting on a particle is factorable into one of the following forms:
(a) $F(x_i, t) = f(x_i)g(t)$ **(b)** $F(\dot{x}_i, t) = f(\dot{x}_i)g(t)$ **(c)** $F(x_i, \dot{x}_i) = f(x_i)g(\dot{x}_i)$
For which cases are the equations of motion integrable?

2-2. A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $\mathbf{F}(\theta, \phi)$. Write the equation of motion.

2-3. If a projectile is fired from the origin of the coordinate system with an initial velocity v_0 and in a direction making an angle α with the horizontal, calculate the time required for the projectile to cross a line passing through the origin and making an angle $\beta < \alpha$ with the horizontal.

2-4. A clown is juggling four balls simultaneously. Students use a video tape to determine that it takes the clown 0.9 s to cycle each ball through his hands (including catching, transferring, and throwing) and to be ready to catch the next ball. What is the minimum vertical speed the clown must throw up each ball?

2-5. A jet fighter pilot knows he is able to withstand an acceleration of $9g$ before blacking out. The pilot points his plane vertically down while traveling at Mach 3 speed and intends to pull up in a circular maneuver before crashing into the ground. **(a)** Where does the maximum acceleration occur in the maneuver? **(b)** What is the minimum radius the pilot can take?

2-6. In the blizzard of '88, a rancher was forced to drop hay bales from an airplane to feed her cattle. The plane flew horizontally at 160 km/hr and dropped the bales from a height of 80 m above the flat range. **(a)** She wanted the bales of hay to land 30 m behind the cattle so as to not hit them. Where should she push the bales out of the airplane? **(b)** To not hit the cattle, what is the largest time error she could make while pushing the bales out of the airplane? Ignore air resistance.

2-7. Include air resistance for the bales of hay in the previous problem. A bale of hay has a mass of about 30 kg and an average area of about 0.2 m^2. Let the resistance be proportional to the square of the speed and let $c_W = 0.8$. Plot the trajectories with a computer if the hay bales land 30 m behind the cattle for both including air resistance and not. If the bales of hay were released at the same time in the two cases, what is the distance between landing positions of the bales?

2-8. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

2-9. Consider a projectile fired vertically in a constant gravitational field. For the same initial velocities, compare the times required for the projectile to reach its maximum height **(a)** for zero resisting force, **(b)** for a resisting force proportional to the instantaneous velocity of the projectile.

2-10. Repeat Example 2.4 by performing a calculation using a computer to solve Equation 2.22. Use the following values: $m = 1$ kg, $v_0 = 10$ m/s, $x_0 = 0$, and $k = 0.1\ \text{s}^{-1}$. Make plots of v versus t, x versus t, and v versus x. Compare with the results of Example 2.4 to see if your results are reasonable.

2-11. Consider a particle of mass m whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity (i.e., kmv^2) is encountered, show that the distance s the particle falls in accelerating from v_0 to v_1 is given by

$$s(v_0 \rightarrow v_1) = \frac{1}{2k}\ln\left[\frac{g - kv_0^2}{g - kv_1^2}\right]$$

2-12. A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. Show that if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it returns to the initial position is

$$\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

2-13. A particle moves in a medium under the influence of a retarding force equal to $mk(v^3 + a^2v)$, where k and a are constants. Show that for any value of the initial speed the particle will never move a distance greater than $\pi/2ka$ and that the particle comes to rest only for $t \to \infty$.

2-14. A projectile is fired with initial speed v_0 at an elevation angle of α up a hill of slope $\beta(\alpha > \beta)$.
(a) How far up the hill will the projectile land?
(b) At what angle α will the range be a maximum?
(c) What is the maximum range?

2-15. A particle of mass m slides down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d after starting from rest is

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg \sin\theta}}$$

where θ is the angle of inclination of the plane.

2-16. A particle is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and find the time required for the particle to return to its starting position. Find the time for $v_0 = 2.4$ m/s and $\alpha = 26°$.

2-17. A strong softball player smacks the ball at a height of 0.7 m above home plate. The ball leaves the player's bat at an elevation angle of 35° and travels toward a fence 2 m high and 60 m away in center field. What must the initial speed of the softball be to clear the center field fence? Ignore air resistance.

2-18. Include air resistance proportional to the square of the ball's speed in the previous problem. Let the drag coefficient be $c_W = 0.5$, the softball radius be 5 cm and the mass be 200 g. **(a)** Find the initial speed of the softball needed now to clear the fence. **(b)** For this speed, find the initial elevation angle that allows the ball to most easily clear the fence. By how much does the ball now vertically clear the fence?

2-19. If a projectile moves such that its distance from the point of projection is always increasing, find the maximum angle above the horizontal with which the particle could have been projected. (Assume no air resistance.)

2-20. A gun fires a projectile of mass 10 kg of the type to which the curves of Figure 2-3 apply. The muzzle velocity is 140 m/s. Through what angle must the barrel be elevated to hit a target on the same horizontal plane as the gun and 1000 m away? Compare the results with those for the case of no retardation.

2-21. Show directly that the time rate of change of the angular momentum about the origin for a projectile fired from the origin (constant g) is equal to the moment of force (or torque) about the origin.

2-22. The motion of a charged particle in an electromagnetic field can be obtained from the **Lorentz equation*** for the force on a particle in such a field. If the electric field vector is **E** and the magnetic field vector is **B**, the force on a particle of mass m that

*See, for example, Heald and Marion, *Classical Electromagnetic Radiation* (95, Section 1.7).

carries a charge q and has a velocity $\mathbf{v}$ is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

where we assume that $v \ll c$(speed of light).

(a) If there is no electric field and if the particle enters the magnetic field in a direction perpendicular to the lines of magnetic flux, show that the trajectory is a circle with radius

$$r = \frac{mv}{qB} = \frac{v}{\omega_c}$$

where $\omega_c \equiv qB/m$ is the *cyclotron frequency.*

(b) Choose the z-axis to lie in the direction of $\mathbf{B}$ and let the plane containing $\mathbf{E}$ and $\mathbf{B}$ be the yz–plane. Thus

$$\mathbf{B} = B\mathbf{k}, \quad \mathbf{E} = E_y\mathbf{j} + E_z\mathbf{k}$$

Show that the z component of the motion is given by

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m}t^2$$

where

$$z(0) \equiv z_0 \quad \text{and} \quad \dot{z}(0) \equiv \dot{z}_0$$

(c) Continue the calculation and obtain expressions for $\dot{x}(t)$ and $\dot{y}(t)$. Show that the time averages of these velocity components are

$$\langle \dot{x} \rangle = \frac{E_y}{B}, \quad \langle \dot{y} \rangle = 0$$

(Show that the motion is periodic and then average over one complete period.)

(d) Integrate the velocity equations found in (c) and show (with the initial conditions $x(0) = -A/\omega_c$, $\dot{x}(0) = E_y/B$, $y(0) = 0$, $\dot{y}(0) = A$) *that*

$$x(t) = \frac{-A}{\omega_c}\cos \omega_c t + \frac{E_y}{B}t, \quad y(t) = \frac{A}{\omega_c}\sin \omega_c t$$

These are the parametric equations of a trochoid. Sketch the projection of the trajectory on the xy–plane for the cases (i) $A > |E_y/B|$, (ii) $A < |E_y/B|$, and (iii) $A = |E_y/B|$. (The last case yields a cycloid.)

2-23. A particle of mass $m = 1$ kg is subjected to a one-dimensional force $F(t) = kte^{-\alpha t}$, where $k = 1$ N/s and $\alpha = 0.5\ \text{s}^{-1}$. If the particle is initially at rest, calculate and plot with the aid of a computer the position, speed, and acceleration of the particle as a function of time.

2-24. A skier weighing 90 kg starts from rest down a hill inclined at 17°. He skis 100 m down the hill and then coasts for 70 m along level snow until he stops. Find the coefficient of kinetic friction between the skis and the snow. What velocity does the skier have at the bottom of the hill?

2-25. A block of mass $m = 1.62$ kg slides down a frictionless incline (Figure 2-A). The block is released a height $h = 3.91$ m above the bottom of the loop.

(a) What is the force of the inclined track on the block at the bottom (point A)?
(b) What is the force of the track on the block at point B?
(c) At what speed does the block leave the track?
(d) How far away from point A does the block land on level ground?
(e) Sketch the potential energy $U(x)$ of the block. Indicate the total energy on the sketch.

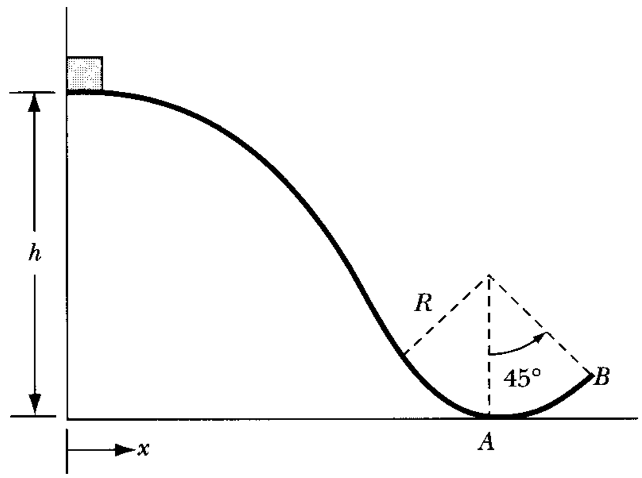

FIGURE 2-A Problem 2-25.

2-26. A child slides a block of mass 2 kg along a slick kitchen floor. If the initial speed is 4 m/s and the block hits a spring with spring constant 6 N/m, what is the maximum compression of the spring? What is the result if the block slides across 2 m of a rough floor that has $\mu_k = 0.2$?

2-27. A rope having a total mass of 0.4 kg and total length 4 m has 0.6 m of the rope hanging vertically down off a work bench. How much work must be done to place all the rope on the bench?

2-28. A superball of mass M and a marble of mass m are dropped from a height h with the marble just on top of the superball. A superball has a coefficient of restitution of nearly 1 (i.e., its collision is essentially elastic). Ignore the sizes of the superball and marble. The superball collides with the floor, rebounds, and smacks the marble, which moves back up. How high does the marble go if all the motion is vertical? How high does the superball go?

2-29. An automobile driver traveling down an 8% grade slams on his brakes and skids 30 m before hitting a parked car. A lawyer hires an expert who measures the coefficient of kinetic friction between the tires and road to be $\mu_k = 0.45$. Is the lawyer correct to accuse the driver of exceeding the 25-MPH speed limit? Explain.

2-30. A student drops a water-filled balloon from the roof of the tallest building in town trying to hit her roommate on the ground (who is too quick). The first student ducks back but hears the water splash 4.021 s after dropping the balloon. If the speed of sound is 331 m/s, find the height of the building, neglecting air resistance.

2-31. In Example 2.10, the initial velocity of the incoming charged particle had no component along the x-axis. Show that, even if it had an x component, the subsequent motion of the particle would be the same—that only the radius of the helix would be altered.

2-32. Two blocks of unequal mass are connected by a string over a smooth pulley (Figure 2-B). If the coefficient of kinetic friction is μ_k, what angle θ of the incline allows the masses to move at a constant speed?

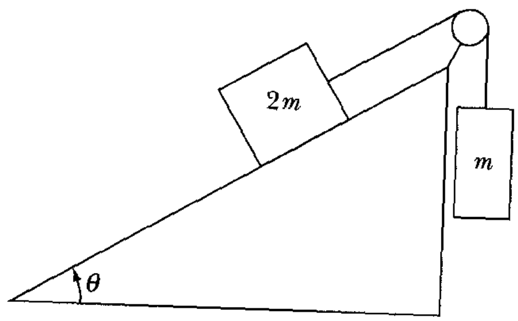

FIGURE 2-B Problem 2-32.

2-33. Perform a computer calculation for an object moving vertically in air under gravity and experiencing a retarding force proportional to the square of the object's speed (see Equation 2.21). Use variables m for mass and r for the object's radius: All the objects are dropped from rest from the top of a 100-m-tall building. Use a value of $c_W = 0.5$ and make computer plots of height y, speed v, and acceleration a versus t for the following conditions and answer the questions:

(a) A baseball of $m = 0.145$ kg and $r = 0.0366$ m.

(b) A ping-pong ball of $m = 0.0024$ kg and $r = 0.019$ m.

(c) A raindrop of $r = 0.003$ m.

(d) Do all the objects reach their terminal speeds? Discuss the values of the terminal velocities and explain their differences.

(e) Why can a baseball be thrown farther than a ping-pong ball even though the baseball is so much more massive?

(f) Discuss the terminal speeds of big and small raindrops. What are the terminal speeds of raindrops having radii 0.002 m and 0.004 m?

2-34. A particle is released from rest ($y = 0$) and falls under the influence of gravity and air resistance. Find the relationship between v and the distance of falling y when the air resistance is equal to **(a)** αv and **(b)** βv^2.

2-35. Perform the numerical calculations of Example 2.7 for the values given in Figure 2-8. Plot both Figures 2-8 and 2-9. Do not duplicate the solution in Appendix H; compose your own solution.

2-36. A gun is located on a bluff of height h overlooking a river valley. If the muzzle velocity is v_0, find the expression for the range as a function of the elevation angle of the gun. Solve numerically for the maximum range out into the valley for a given h and v_0.

2-37. A particle of mass m has speed $v = \alpha/x$, where x is its displacement. Find the force $F(x)$ responsible.

2-38. The speed of a particle of mass m varies with the distance x as $v(x) = \alpha x^{-n}$. Assume $v(x = 0) = 0$ at $t = 0$. **(a)** Find the force $F(x)$ responsible. **(b)** Determine $x(t)$ and **(c)** $F(t)$.

2-39. A boat with initial speed v_0 is launched on a lake. The boat is slowed by the water by a force $F = -\alpha e^{\beta v}$. **(a)** Find an expression for the speed $v(t)$. **(b)** Find the time and **(c)** distance for the boat to stop.

2-40. A particle moves in a two-dimensional orbit defined by

$$x(t) = A(2\alpha t - \sin \alpha t)$$
$$y(t) = A(1 - \cos \alpha t)$$

(a) Find the tangential acceleration a_t and normal acceleration a_n as a function of time where the tangential and normal components are taken with respect to the velocity.
(b) Determine at what times in the orbit a_n has a maximum.

2-41. A train moves along the tracks at a constant speed u. A woman on the train throws a ball of mass m straight ahead with a speed v with respect to herself. **(a)** What is the kinetic energy gain of the ball as measured by a person on the train? **(b)** by a person standing by the railroad track? **(c)** How much work is done by the woman throwing he ball and **(d)** by the train?

2-42. A solid cube of uniform density and sides of b is in equilibrium on top of a cylinder of radius R (Figure 2-C). The planes of four sides of the cube are parallel to the axis of the cylinder. The contact between cube and sphere is perfectly rough. Under what conditions is the equilibrium stable or not stable?

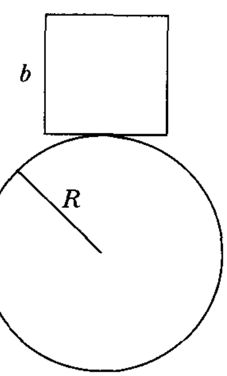

FIGURE 2-C Problem 2-42.

2-43. A particle is under the influence of a force $F = -kx + kx^3/\alpha^2$, where k and α are constants and k is positive. Determine $U(x)$ and discuss the motion. What happens when $E = (1/4)k\alpha^2$?

2-44. Solve Example 2.12 by using forces rather than energy. How can you determine whether the system equilibrium is stable or unstable?

2-45. Describe how to determine whether an equilibrium is stable or unstable when $(d^2U/dx^2)_0 = 0$.

2-46. Write the criteria for determining whether an equilibrium is stable or unstable when all derivatives up through order n, $(d^nU/dx^n)_0 = 0$.

2-47. Consider a particle moving in the region $x > 0$ under the influence of the potential

$$U(x) = U_0\left(\frac{a}{x} + \frac{x}{a}\right)$$

where $U_0 = 1$ J and $a = 2$ m. Plot the potential, find the equilibrium points, and determine whether they are maxima or minima.

2-48. Two gravitationally bound stars with equal masses m, separated by a distance d, revolve about their center of mass in circular orbits. Show that the period τ is proportional to $d^{3/2}$ (Kepler's Third Law) and find the proportionality constant.

2-49. Two gravitationally bound stars with unequal masses m_1 and m_2, separated by a distance d, revolve about their center of mass in circular orbits. Show that the period τ is proportional to $d^{3/2}$ (Kepler's Third Law) and find the proportionality constant.

2-50. According to special relativity, a particle of rest mass m_0 accelerated in one dimension by a force F obeys the equation of motion $dp/dt = F$. Here $p = m_0v/(1 - v^2/c^2)^{1/2}$ is the relativistic momentum, which reduces to m_0v for $v^2/c^2 \ll 1$. **(a)** For the case of constant F and initial conditions $x(0) = 0 = v(0)$, find $x(t)$ and $v(t)$. **(b)** Sketch your result for $v(t)$. **(c)** Suppose that $F/m_0 = 10$ m/s² ($\approx g$ on Earth). How much time is required for the particle to reach half the speed of light and of 99% the speed of light?

2-51. Let us make the (unrealistic) assumption that a boat of mass m gliding with initial velocity v_0 in water is slowed by a viscous retarding force of magnitude bv^2, where b is a constant. **(a)** Find and sketch $v(t)$. How long does it take the boat to reach a speed of $v_0/1000$? **(b)** Find $x(t)$. How far does the boat travel in this time? Let $m =$ 200 kg, $v_0 = 2$ m/s, and $b = 0.2$ Nm^{-2}s^2.

2-52. A particle of mass m moving in one dimension has potential energy $U(x) = U_0[2(x/a)^2 - (x/a)^4]$, where U_0 and a are positive constants. **(a)** Find the force $F(x)$, which acts on the particle. **(b)** Sketch $U(x)$. Find the positions of stable and unstable equilibrium. **(c)** What is the angular frequency ω of oscillations about the point of stable equilibrium? **(d)** What is the minimum speed the particle must have at the origin to escape to infinity? **(e)** At $t = 0$ the particle is at the origin and its velocity is positive and equal in magnitude to the escape speed of part **(d)**. Find $x(t)$ and sketch the result.

2-53. Which of the following forces are conservative? If conservative, find the potential energy $U(\mathbf{r})$. **(a)** $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$. **(b)** $F_x = -ze^{-x}$, $F_y = \ln z$, $F_z = e^{-x} + y/z$. **(c)** $\mathbf{F} = \mathbf{e}_r a/r$ (a, b, c are constants).

2-54. A potato of mass 0.5 kg moves under Earth's gravity with an air resistive force of $-kmv$. **(a)** Find the terminal velocity if the potato is released from rest and $k =$ 0.01 s^{-1}. **(b)** Find the maximum height of the potato if it has the same value of k, but it is initially shot directly upward with a student-made potato gun with an initial velocity of 120 m/s.

2-55. A pumpkin of mass 5 kg shot out of a student-made cannon under air pressure at an elevation angle of 45° fell at a distance of 142 m from the cannon. The students used light beams and photocells to measure the initial velocity of 54 m/s. If the air resistive force was $F = -kmv$, what was the value of k?